Tutorial 10

In the following problems, V denotes a finite-dimensional vector space.

1. Suppose V is a real inner product space and $T \in \mathcal{L}(V)$ is an isometry such that $T^2 - I$ is invertible. Show that $\dim V$ is even.

2. Suppose V is a real inner product space and $T \in \mathcal{L}(V)$ is an isometry. Must there exist $n \in \mathbb{N}$ such that $T^n = I$?

3. Suppose $\mathbb{F} = \mathbb{R}$ and $f \in \mathbb{R}[x]$ has no real roots. Let $T \in \mathcal{L}(V)$ be such that $f(T)$ is nilpotent.

   (a) Show that $\dim V$ is even.

   (b) Further suppose $\deg f = 2$. Show that $(f(T))^{n/2} = 0$, where $n = \dim V$.

4. Let V be a real inner product space and $R \in \mathcal{L}(V)$ be an isometry.

   (a) We say R is a *reflection* if

   $$V = E(1, R) \oplus E(-1, R)$$

   with $\dim E(-1, R) = 1$. Classify all reflections of V.

   (b) We say R is a *rotation* if $\det R = 1$. Show that R cannot be both a reflection and a rotation.

   (c) Classify all rotations of V.

5. Recall that given vector spaces V and W over the same field $\mathbb{F}$ we can (roughly) define the *tensor product* $V \otimes W$ as the set of formal linear combinations $v \otimes w$ with $v \in V$ and $w \in W$. We require the tensor product to be bilinear, i.e.

   $$(cv_1 + v_2) \otimes w = c(v_1 \otimes w) + v_2 \otimes w \quad \text{and} \quad v \otimes (cw_1 + w_2) = c(v \otimes w_1) + v \otimes w_2$$

   for all $c \in \mathbb{F}$, $v_1, v_2, v \in V$, and $w_1, w_2, w \in W$.

   We may use without proof that if $\{e_1, \ldots, e_k\}$ is a basis for V and $\{f_1, \ldots, f_n\}$ is a basis for W then $\{e_i \otimes f_j \in V \otimes W : 1 \leq i \leq k, 1 \leq j \leq n\}$ is a basis for $V \otimes W$.

   Let V be a real vector space. Considering $\mathbb{C}$ as a two-dimensional real vector space we define the real vector space $V^{\mathbb{C}}$ as

   $$V^{\mathbb{C}} = V \otimes \mathbb{C}$$

(a) Suppose we wish to make $V^{\mathbb{C}}$ a complex vector space. How should we define complex scalar multiplication on $V^{\mathbb{C}}$?

(b) As a complex vector space, what is $V^{\mathbb{C}}$?

(c) Now suppose V is a vector space over $\mathbb{Q}$. Inspired by the above, how can we "extend" V to a real vector space?

6. Suppose V is a real vector space and let $J \in \mathcal{L}(V)$ be such that $J^2 = -I$. We call such J a *complex structure* on V.

   (a) Show that only even-dimensional spaces have complex structures.

   (b) Use J to define complex scalar multiplication on V, turning V into a complex vector space ${}_{\mathbb{C}}V$.

   (c) What is $\dim {}_{\mathbb{C}}\mathbb{R}^2$?

   (d) What is $\dim {}_{\mathbb{C}}\mathbb{R}^{2n}$?